Microeconomics II Mock Exam

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Exercise 1

For the following game find all Nash equilibria and indicate which are THP and why.

P1 \ P2	LL	L	M	R
U	10,3	-10,2	0,0	-10,-10
D	-10,-10	10,-5	2,0	10,3

We look at best responses to find all pairs (a_1, a_2) such that $a_1 \in BR_1(a_2)$, and $a_2 \in BR_2(a_1)$.

$$BR_1(LL) = U \;\; BR_1(L) = D \;\; BR_1(M) = D \;\; BR_1(R) = D$$
$$BR_2(U) = LL \;\; BR_2(D) = R$$

Based on the BRs, there are two PSNE = (U,LL),(D,R). Now let's consider mixed strategies and denote: $\sigma_2(LL) = \alpha_1, \sigma_2(L) = \alpha_2, \sigma_2(M) = \alpha_3, \sigma_2(R) = (1 - \sum_{i=1}^{3} \alpha_i)$. Player 1 is indifferent between strategies U and D when:

$$10\alpha_1 - 10\alpha_2 - 10(1 - \sum_{i=1}^{3} \alpha_i) = -10\alpha_1 + 10\alpha_2 + 2\alpha_3 + 10(1 - \sum_{i=1}^{3} \alpha_i)$$
$$\implies 20\alpha_1 + 9\alpha_3 = 10$$

Now denote $\sigma_1(U) = \beta_1, \sigma_2(D) = 1-\beta_1$ and consider indifference conditions for player 2:

$$\text{P2 indifferent between LL and L:}$$
$$3\beta_1 - 10(1-\beta_1) = 2\beta_1 - 5(1-\beta_1)$$
$$\beta_1 = \frac{5}{6}$$
$$\text{P2 indifferent between L and M:}$$
$$2\beta_1 - 5(1-\beta_1) = 0$$
$$\beta_1 = \frac{5}{7}$$
$$\text{P2 indifferent between M and R:}$$
$$-10\beta_1 + 3(1-\beta_1) = 0$$
$$\beta_1 = \frac{3}{13}$$
$$\text{P2 indifferent between LL and R:}$$
$$3\beta_1 - 10(1-\beta_1) = -10\beta_1 + 3(1-\beta_1)$$
$$\beta_1 = \frac{1}{2}$$
$$\text{P2 indifferent between L and R:}$$
$$2\beta_1 - 5(1-\beta_1) = -10\beta_1 + 3(1-\beta_1)$$
$$\beta_1 = \frac{2}{5}$$

Given the above indifference conditions we can derive the best responses for player two:

$$BR_2(\beta_1) = \begin{cases} R\ if\ \beta_1 < 3/13 \\ \Delta\{M, R\}\ if\ \beta_1 = 3/13 \\ M\ if\ \beta_1 \in (3/13, 5/7) \\ \Delta\{L, M\}\ if\ \beta_1 = 5/7 \\ L\ if\ \beta_1 \in (5/7, 5/6) \\ \Delta\{LL, L\}\ if\ \beta_1 = 5/6 \\ LL\ if\ \beta_1 > 5/6 \end{cases}$$

Now notice that:

$$BR_1(\sigma_2(.)) = \begin{cases} D\ if\ \sigma_2(R) = 1 \\ D\ if\ \sigma_2(M) = 1 \\ D\ if\ \sigma_2(L) = 1 \\ U\ if\ \sigma_2(LL) = 1 \end{cases}$$

We thus have the following set of MSNE:

$$MSNE = \{(\sigma_1(U) = 3/13, \sigma_2(M) + \sigma_2(R) = 1),$$
$$(\sigma_1(U) = 5/7, \sigma_2(M) + \sigma_2(L) = 1), (\sigma_1(U) = 5/6, \sigma_2(L) + \sigma_2(LL) = 1),$$
$$(\sigma_1(U) \in [0,1], 20\sigma_2(LL) + 9\sigma_2(M) = 10), PSNE\}$$

Now let's see whether the PSNE is a THP. Add a small tremble to probability of playing LL by P2 so that we have: $\sigma_2(LL) = 1-\epsilon_2, \sigma_2(L) = \sigma_2(M) = \sigma_2(R) = \frac{\epsilon_2}{3}$. For which value of ϵ_2 playing U by player 1 remains a best response?

$$u_1(U) = 10(1-\epsilon_2) - 10\frac{\epsilon_2}{3} - 10\frac{\epsilon_2}{3} = 10 - \frac{50}{3}\epsilon_2$$
$$u_1(D) = -10(1-\epsilon_2) + 10\frac{\epsilon_2}{3}) + 2\frac{\epsilon_2}{3} + 10\frac{\epsilon_2}{3} = -10 + \frac{52}{3}\epsilon_2$$
$$10 - \frac{50}{3}\epsilon_2 > -10 + \frac{52}{3}\epsilon_2$$
$$\implies \epsilon_2 < \frac{10}{17}$$

Now add a small tremble to probability of playing U by P1 so that we have: $\sigma_1(U) = 1-\epsilon_1, \sigma_2(D) = \epsilon_1$. For which value of ϵ_1 playing LL by player 2 remains a best response?

$$u_2(LL) = 3(1-\epsilon_1) - 10\epsilon_1 = 3 - 13\epsilon_1$$
$$u_2(L) = 2(1-\epsilon_1) - 5\epsilon_1 = 2 - 7\epsilon_1$$
$$u_2(M) = 0$$
$$u_2(R) = -10(1-\epsilon_1) + 3\epsilon_1 = -10 + 13\epsilon_1$$
$$3 - 13\epsilon_1 > 2 - 7\epsilon_1$$
$$\implies \epsilon_1 < \frac{1}{6}$$
$$3 - 13\epsilon_1 > 0$$
$$\epsilon_1 < \frac{3}{13}$$
$$3 - 13\epsilon_1 > -10 + 13\epsilon_1$$
$$\implies \epsilon_1 < \frac{1}{2}$$

Hence, LL remains a BR for player 2 if $\epsilon_1 < \frac{1}{6}$. We conclude that (U,LL) is a THP Nash equilibrium. An example of converging sequence might be: $\sigma_1^k = (1-(\frac{1}{7})^k, (\frac{1}{7})^k), \sigma_2^k = (1-(\frac{10}{18})^k, (\frac{10}{54})^k, (\frac{10}{54})^k, (\frac{10}{54})^k)$.
What about the PSNE (D,R)? We repeat the same steps.
Now let's see whether the PSNE is a THP. Add a small tremble to probability of playing R by P2 so that we have: $\sigma_2(R) = 1-\epsilon_2, \sigma_2(L) = \sigma_2(M) = \sigma_2(LL) = \frac{\epsilon_2}{3}$. For which value of ϵ_2 playing D by player 1 remains a best response?

$$u_1(U) = 10\frac{\epsilon_2}{3} - 10\frac{\epsilon_2}{3} - 10(1-\epsilon_2) = -10 + 10\epsilon_2$$
$$u_1(D) = -10\frac{\epsilon_2}{3} + 10\frac{\epsilon_2}{3} + 2\frac{\epsilon_2}{3} + 10(1-\epsilon_2) = 10 - \frac{28}{3}\epsilon_2$$
$$10 - \frac{28}{3}\epsilon_2 > -10 + 10\epsilon_2$$
$$\implies \epsilon_2 < \frac{30}{29}$$

Hence, for all possible values of tremble player 1 would not change her strategy. Now add a small tremble to probability of playing D by P1 so that we have: $\sigma_1(D) = 1-\epsilon_1, \sigma_2(U) = \epsilon_1$. For which value of ϵ_1 playing

R by player 2 remains a best response?

$$
\begin{aligned}
u_2(LL) &= 3\epsilon_1 - 10(1-\epsilon_1) = -10 + 13\epsilon_1 \\
u_2(L) &= 2\epsilon_1 - 5(1-\epsilon_1) = 7\epsilon_1 - 5 \\
u_2(M) &= 0 \\
u_2(R) &= -10\epsilon_1 + 3(1-\epsilon_1) = 3 - 13\epsilon_1 \\
-13\epsilon_1 + 3 &> -10 + 13\epsilon_1 \\
\implies \epsilon_1 &< \frac{1}{2} \\
3 - 13\epsilon_1 &> 7\epsilon_1 - 5 \\
\implies \epsilon_1 &< \frac{2}{5} \\
3 - 13\epsilon_1 &> 0 \\
\implies \epsilon_1 &< \frac{3}{13}
\end{aligned}
$$

Hence, we conclude that player 2 will not change her strategy if the value of tremble $\epsilon < \frac{3}{13}$.
Both PSNE are THP. This is not surprising, given that none of the strategies in the Nash equilibrium is weakly dominated and the game has MSNE.

Exercise 2

Consider the three-player game with the payoffs given in tables A, B and C. Player 1 chooses rows, player 2 chooses columns, and Player 3 chooses one of the three tables.[1]

1. *What are the pure strategy equilibrium payoffs?*
2. *Show that there is a correlated equilibrium in which player 3 chooses B and players 1 and 2 play (U,L) and (D,R) with equal probabilities.*

	L	R
U	1,1,4	1,1,1
D	2,1,1	1,1,1

(a) Table A

	L	R
U	3,3,3	1,1,1
D	1,1,1	3,3,3

(b) Table B

	L	R
U	1,1,1	1,1,1
D	1,2,1	1,1,4

(c) Table C

To answer the first question we look for best response of each player given each strategy played by her

[1] This exercise is a modification of one of exercises included in "A Course in Game Theory", by A. Rubinstein and M.J. Osborne (1994)

opponents:

$$BR_1(a_2 = L, a_3 = A) = D \;\; BR_1(a_2 = R, a_3 = A) = \Delta\{U, D\}$$
$$BR_1(a_2 = L, a_3 = B) = U \;\; BR_1(a_2 = R, a_3 = B) = D$$
$$BR_1(a_2 = L, a_3 = C) = \Delta\{U, D\} \;\; BR_1(a_2 = R, a_3 = C) = \Delta\{U, D\}$$
$$BR_2(a_1 = U, a_3 = A) = \Delta\{L, R\} \;\; BR_2(a_1 = D, a_3 = A) = \Delta\{L, R\}$$
$$BR_2(a_1 = U, a_3 = B) = L \;\; BR_2(a_1 = D, a_3 = B) = R$$
$$BR_2(a_1 = U, a_3 = C) = \Delta\{L, R\} \;\; BR_2(a_1 = D, a_3 = C) = L$$
$$BR_3(a_1 = U, a_2 = L) = A \;\; BR_3(a_1 = U, a_2 = R) = \Delta\{A, B, C\}$$
$$BR_3(a_1 = D, a_2 = L) = \Delta\{A, B, C\} \;\; BR_3(a_1 = D, a_2 = R) = C$$

Hence, we have the following set of PSNE: {(U,R,A),(D,L,A),(D,L,C),(U,R,C)} yielding the payoffs: {(1,1,1),(2,1,1),(1,2,1)}. In order to define the correlated equilibrium we need:

1. Set of outcomes and associated probability measure:

$$\Omega = \{ULB, DRB\}$$
$$\pi(ULB) = \pi(DRB) = 1/2$$

2. Partition sets:

$$\mathscr{P}_1 = \mathscr{P}_2 = \{ULB\}, \{DRB\}$$
$$\mathscr{P}_3 = \Omega$$

3. Strategies each player would employ following a message:

$$\sigma_1(ULB) = U \;\; \sigma_1(DRB) = D$$
$$\sigma_2(ULB) = L \;\; \sigma_2(DRB) = R$$
$$\sigma_3(\Omega) = B$$

Let's check whether it is indeed an equilibrium. We start with checking the best response of player 1 after receiving each signal.

- The correlated device draws $\{ULB\}$

$$Pr(ULB|ULB) = 1$$
$$u_1(U|ULB) = 3 \;\; u_1(D|ULB) = 1$$
$$\implies BR_1(ULB) = U$$

- The correlated device draws $\{DRB\}$:

$$Pr(DRB|DRB) = 1$$
$$u_1(D|DRB) = 3 \;\; u_1(U|DRB) = 1$$
$$\implies BR_1(DRB) = D$$

Player 1 plays accordingly to the cue sent by the device. Let's now consider player 2.

- The correlated device draws $\{ULB\}$:

$$Pr(ULB|ULB) = 1$$
$$u_2(L|ULB) = 3 \;\; u_2(R|ULB) = 1$$
$$\implies BR_2(ULB) = L$$

- The correlated device draws $\{DRB\}$:

$$Pr(DRB|DRB) = 1$$
$$u_2(R|DRB) = 3 \;\; u_2(L|DRB) = 1$$
$$\implies BR_2(DRB) = R$$

Player 2 plays accordingly to the cue sent by the device. Let's now consider player 3.

- The correlated device draws $\{ULB\}$:

$$Pr(ULB|B) = \frac{1 * 1/2}{1} = 1/2$$
$$u_3(B|ULB) = 3\frac{1}{2} + 3\frac{1}{2} = 3$$
$$u_3(A|ULB) = 4\frac{1}{2} + 1\frac{1}{2} = 2.5$$
$$u_3(C|ULB) = 1\frac{1}{2} + 4\frac{1}{2} = 2.5$$
$$\implies BR_3(ULB) = B$$

- The correlated device draws $\{DRB\}$:

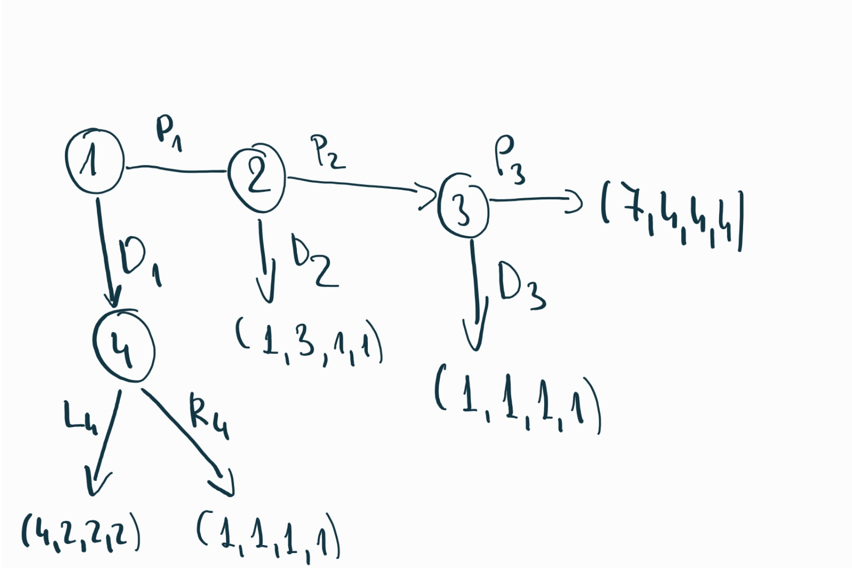
$$Pr(DRB|B) = \frac{1 * 1/2}{1} = 1/2$$
$$u_3(B|DRB) = 3\frac{1}{2} + 3\frac{1}{2} = 3$$
$$u_3(A|DRB) = 4\frac{1}{2} + 1\frac{1}{2} = 2.5$$
$$u_3(C|DRB) = 1\frac{1}{2} + 4\frac{1}{2} = 2.5$$
$$\implies BR_3(DRB) = B$$

Player 3 plays accordingly to the cue sent by the device.
Hence, we have constructed a correlated equilibrium in which player chooses B and players 1 and 2 play (U,L) and (D,R) with equal probabilities.

Exercise 3

Consider the following four-player game:

1. *What payoffs are possible in pure strategy Nash equilibria?*



2. *What payoffs are possible in a sequential equilibrium?*

3. *Is it possible to construct a self-confirming equilibrium with the following strategy profile:* $\sigma = [(0.5D_1, 0.5P_1);(D_2);(P_3);(L_4)]$*? If yes, what beliefs we would need to construct? If not, how would you change the strategies of the other players so that we still have P2 playing drop?*

In order to find the PSNE we derive the normal form of the game. Note that player 2 chooses rows and player 3 chooses columns.

	P_3	D_3
P_2	4,2,2,2	4,2,2,2
D_2	4,2,2,2	4,2,2,2

(a) P1 chooses D_1, P4 chooses L_4

	P_3	D_3
P_2	1,1,1,1	1,1,1,1
D_2	1,1,1,1	1,1,1,1

(b) P1 chooses D_1, P4 chooses R_4

	P_3	D_3
P_2	7,4,4,4	1,1,1,1
D_2	1,3,1,1	1,3,1,1

(c) P1 chooses P_1, P4 chooses L_4 or R_4

After choosing best responses of each player to each of the strategies we get the following PSNE:
$\{(D_1, P_2, D_3, L_4),(D_1, D_2, P_3, L_4),(D_1, D_2, D_3, L_4),(P_1, P_2, P_3, L_4),(P_1, P_2, P_3, R_4)\}$.
Hence, the possible payoffs in the PSNE are (4,2,2,2) and (7,4,4,4).

To solve the sequential equilibrium we use backward induction. If we start from player 4, we know that he would choose L_4. And if we start from player 3, we know that she would choose P_3. Knowing that, P2 would choose P_2. Knowing what P4, P3 and P2 would choose, P1 would choose P1. Hence, the only possible payoff in the sequential equilibrium is (7,4,4,4). Beliefs are consistent with the strategies (note that on- and off-the path), and strategies with the beliefs.

What beliefs we would need to construct a self-confirming equilibrium with the profile of strategies $\sigma = [(0.5D_1, 0.5P_1);(D_2);(P_3);(L_4)]$?
We would need P2 to believe that P3 would drop, so she would drop too. Also, we would need to construct a belief for two types of P1 (let's denote them A and B). The type A believes (correctly) that P2 would drop, so he drops too. The type B of P1 believes that P2 would pass with some probability which would induce him to pass. In this solution I would assume that P1 believes that P2 would pass with probability equal

to 0.6.

Therefore, we can construct a following heterogeneous self-confirming equilibrium:

- P1 is of Type A:

$$\mu_1^A[\Pr(D_2) = 1, \Pr(D_3) = 0, \Pr(L_4) = 1] = 1$$
$$u_1^A(D_1, \sigma_{-i}) = 4 > 1 = u_1^A(P_1, \sigma_{-i})$$

- P1 is of Type B:

$$\mu_1^B[\Pr(D_2) = 0.4, \Pr(D_3) = 0, \Pr(L_4) = 1] = 1$$
$$u_1^B(D_1, \{0.4D_2 + 0.5P_2; 0D_3; L_4\}) = 4 < 0.4 * 1 + 0.6 * 7 = 4.6 = u_1^A(P_1, \{0.4D_2 + 0.5P_2; 0D_3; L_4\})$$

- P2 believes P3 would drop so she drops too.

$$\mu_2[\Pr(D_1) = 1, \Pr(D_3) = 1, \Pr(L_4) = 1] = 1$$
$$u_2(D_2, D_3) = 3 > 1 = u_2(P_1, D_3)$$

- P3 will pass.

$$\mu_3[\Pr(D_1) = 1, \Pr(D_2) = 1, \Pr(L_4) = 1] = 1$$
$$u_3(P_3, \sigma_{-i}) = 4 > 1 = u_3(D_3, \sigma_{-i})$$

- P4 will play left.

$$\mu_4[\Pr(D_1) = 1, \Pr(D_2) = 1, \Pr(P_3) = 1] = 1$$
$$u_4(L_4, \sigma_{-i}) = 4 > 1 = u_4(R_4, \sigma_{-i})$$

We get the following heterogeneous self-confirming equilibrium:

$$\begin{aligned}
(\sigma, \mu) = \{\sigma_1^A = D_1, \sigma_1^B = P_1, \sigma_2 = D_2, \sigma_3 = P_3, \sigma_4 = L_4\},\\
\mu = \{\mu_1^A[\Pr(D_2) = 1, \Pr(D_3) = 0, \Pr(L_4) = 1] = 1, \mu_1^B[\Pr(D_2) = 0.4, \Pr(D_3) = 0, \Pr(L_4) = 1] = 1,\\
\mu_2[\Pr(D_1) = 1, \Pr(D_3) = 1, \Pr(L_4) = 1] = 1,\\
\mu_3[\Pr(D_1) = 1, \Pr(D_2) = 1, \Pr(L_4) = 1] = 1, \mu_4[\Pr(D_1) = 1, \Pr(D_2) = 1, \Pr(P_3) = 1] = 1\}
\end{aligned}$$

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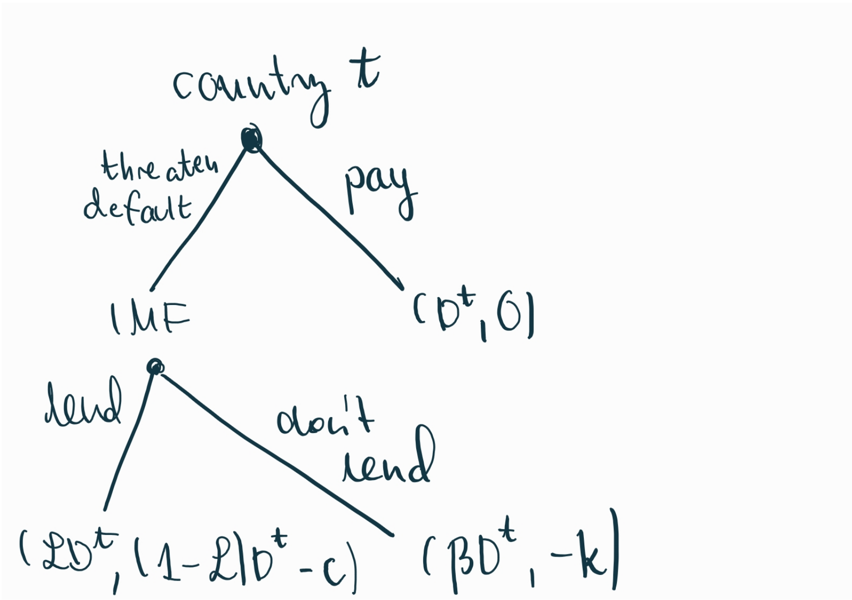
Exercise 4

Suppose that the International Monetary Fund considers the situation of public finance in some two countries sequentially. At date $t \in \{1,2\}$, country t decides whether to pay its debt to the IMF, D^t, or to threaten default.[2] *If it threatens default, the IMF can start another lending program or do nothing. If the IMF does nothing, it results in default for the country in trouble. However, if it starts a program, a country needs to introduce some structural reforms which is costly (cost c). The stage game is illustrate below. Note that $1 > \alpha > 0, \alpha > \beta$ and $k > c > 0$. The management of the IMF can be "soft" or "tough" (never lend). Only the management in the headquarters in DC knows whether it is soft or tough. Assume that the IMF's discount factor is equal to one and that $(1-p)((1-\alpha)D^t - c) - pk > 0$ for $t = 1,2$, where p is the prior probability that the bank is tough.*[3]

1. *Solve for the equilibrium of this two-period game.*
2. *If the IMF had the choice between facing the low-debt country and the high-debt country first, which one would it choose?*



It is a finite horizon game with one long-run player (IMF) and two short-term players (countries in both periods). If payoffs are known to all players, there is a unique sequential equilibrium in which countries in both periods threaten the default and the IMF always lends. Why? In the last period, the IMF always lends, and by backward induction it also lends in the previous period.

However, a more interesting case is when we assume that the value of β is private information and hence we can introduce reputation effects to the model.

[2] This is not very realistic as debts to the IMF are paid in priority by governments.

[3] This exercise is a modification of an exercise included in "Game Theory" by D. Fudenberg and J. Tirole (1991)

Country t (ct) will threaten the default if:

$$E(u_{ct}(TD, BR_{IMF}(TD)) > E(u_{ct}(P))$$
$$(1-\mu_{ct}(T)((1-\alpha)D^t - c) - \mu_{ct}(T)k > 0$$
$$\implies D^t > \frac{\mu_{ct}(T)(k-c)+c}{(1-\alpha)(1-\mu_{ct}(T))}$$

Where $\mu_{ct}(T)$ is the probability the country in period t assigns to the IMF being tough. If the inequality is reversed, a country in period t will not threaten default (in the case of the equality a country t would be indifferent, but let's not discuss this case here).
Now, note that when a country threatens the default it is always profitable for the IMF to lend more money (as $\alpha D^t > \beta D^t$). However, we assume it is a private information and then it is a reputation's game and beliefs of a country in the second period are derived from actions of players in the first period.

Let's start from period $t = 2$.

Second period

Country $t = 2$: depending on what the IMF did in the first period it will adjust its' beliefs towards the type of the IMF's management.

1. Case 1: Country $t = 1$ threatened default:

   - if the IMF responded by agreeing to lend: $\sigma_{IMF,t=1} = L$.

     $$\mu_{c2}(T|L_1) = \frac{Pr(L_1|T)p}{Pr(L_1)}$$

     Because $Pr(L_1|T) < Pr(L_1)$ then $\mu_{c2}(T|L_1) < p$.
     Then, the country in $t = 2$ will threaten default if:

     $$D^2 > \frac{\mu_{c2}(T|L_1)(k-c)+c}{(1-\alpha)(1-\mu_{c2}(T|L_1))}$$

     If the inequality is reversed, it will not threaten the default. The response of the IMF would be to lend money, as $u_{IMF,t=2}(L) = \alpha D^2 > u_{IMF,t=2}(DL) = \beta D^2$.

   - if the IMF responded by not lending: $\sigma_{IMF,t=1} = DL$.

     $$\mu_{c2}(T|DL) = \frac{1 * p}{Pr(DL_1)} > p$$

     Then, the country in $t = 2$ will threaten the default if:

     $$D^2 > \frac{\mu_{c2}(T|DL_1)(k-c)+c}{(1-\alpha)(1-\mu_{c2}(T|DL_1))}$$

     If the inequality is reversed, it will not threaten the default. The response of the IMF would be to lend money, as $u_{IMF,t=2}(L) = \alpha D^2 > u_{IMF,t=2}(DL) = \beta D^2$.

2. Case 2: Country $t = 1$ did not threaten default.
   In that case, the posterior belief that the IMF is tough is equal to the prior, $\mu_{c2}(T) = p$ and country $t = 2$ threatens default if:

   $$D^2 > \frac{p(k-c)+c}{(1-\alpha)(1-p)}$$

Note that if the above condition holds and the country $t = 2$ threatens default, it is optimal for the IMF to lend no matter its' type.

First period

1. Case 1: Country $t = 1$ threatens default if $D^1 > \frac{p(k-c)+c}{(1-\alpha)(1-p)}$.

   - IMF does not lend money if:

   $$\mu_{IMF}(c2 = P)E(D^2) + \beta D^1 > (1 - \mu_{IMF}(c2 = P))\alpha E(D^2) + \alpha D^1$$
   $$\implies \mu_{IMF}(c2 = P) > \frac{D^1(\alpha - \beta)/E(D^2) + 1}{1-\alpha}$$

     where $\mu_{IMF}(c2 = P)$ is the probability that the IMF assigns for the country in the second period to pay its' debts. Note that it depends on the expectations of the level of D^2.
   - Otherwise, it lends money to country in $t = 1$ (if the above condition is satisfied with equality, it is indifferent between lending and not lending).

2. Case 2: Country $t = 1$ does not threaten the default if $D^1 < \frac{p(k-c)+c}{(1-\alpha)(1-p)}$.

Hence, depending depending on the value of D^1, D^2 and $E(D^2)$ we can arrive to different equilibrium:

1. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} > D^1$ and $\frac{p(k-c)+c}{(1-\alpha)(1-p)} > D^2$ and $\mu_{IMF}(c2 = P) > \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = DL, \sigma_{IMF,t=2} = L, \sigma_{c1} = P, \sigma_{c2} = P, \mu_{c1}(T) = p, \mu_{IMF}(c2 = P)(T) = p\}$
2. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} > D^1$ and $\frac{p(k-c)+c}{(1-\alpha)(1-p)} > D^2$ and $\mu_{IMF}(c2 = P) < \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = L, \sigma_{IMF,t=2} = L, \sigma_{c1} = P, \sigma_{c2} = P, \mu_{c1}(T) = p, \mu_{c2}(T) = p\}$
3. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} > D^1$ and $\frac{p(k-c)+c}{(1-\alpha)(1-p)} < D^2$ and $\mu_{IMF}(c2 = P) < \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = L, \sigma_{IMF,t=2} = L, \sigma_{c1} = P, \sigma_{c2} = TD, \mu_{c1}(T) = p, \mu_{c2}(T) = p\}$
4. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} > D^1$ and $\frac{p(k-c)+c}{(1-\alpha)(1-p)} < D^2$ and $\mu_{IMF}(c2 = P) > \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = DL, \sigma_{IMF,t=2} = L, \sigma_{c1} = P, \sigma_{c2} = TD, \mu_{c1}(T) = p, \mu_{c2}(T) = p\}$
5. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} < D^1$ and $\frac{\mu_{c2}(T)(k-c)+c}{(1-\alpha)(1-\mu_{c2}(T))} < D^2$ and $\mu_{IMF}(c2 = P) < \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = L, \sigma_{IMF,t=2} = L, \sigma_{c1} = TD, \sigma_{c2} = TD, \mu_{c1}(T) = p, \mu_{c2}(T) < p\}$
6. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} < D^1$ and $\frac{\mu_{c2}(T)(k-c)+c}{(1-\alpha)(1-\mu_{c2}(T))} < D^2$ and $\mu_{IMF}(c2 = P) > \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = DL, \sigma_{IMF,t=2} = L, \sigma_{c1} = TD, \sigma_{c2} = TD, \mu_{c1}(T) = p, \mu_{c2}(T) > p\}$
7. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} < D^1$ and $\frac{\mu_{c2}(T)(k-c)+c}{(1-\alpha)(1-\mu_{c2}(T))} > D^2$ and $\mu_{IMF}(c2 = P) < \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = L, \sigma_{IMF,t=2} = L, \sigma_{c1} = TD, \sigma_{c2} = P, \mu_{c1}(T) = p, \mu_{c2}(T) < p\}$
8. If $\frac{p(k-c)+c}{(1-\alpha)(1-p)} < D^1$ and $\frac{\mu_{c2}(T)(k-c)+c}{(1-\alpha)(1-\mu_{c2}(T))} > D^2$ and $\mu_{IMF}(c2 = P) > \frac{D^1(\alpha-\beta)/E(D^2)+1}{1-\alpha}$:
   $\sigma = \{\sigma_{IMF,t=1} = DL, \sigma_{IMF,t=2} = L, \sigma_{c1} = TD, \sigma_{c2} = P, \mu_{c1}(T) = p, \mu_{c2}(T) > p\}$

It is better for the IMF to face the low-debt country first. Firstly, we know that the probability of a low-debt country paying its' debt is higher. Also, we know that $D^1_{low} < D^2_{high}$. Then, the expected payoff of facing the low-debt country first and then high-debt country is higher than otherwise.
Let's consider several cases:

1. low-debt country pays in the first period and threatens default in the second, then high-debt coun-

try always threatens default, IMF always lends:

$$D^1_{low} + \alpha D^2_{high} > \alpha D^1_{high} + \alpha D^2_{low}$$

2. the first country always threatens default and the second country always pays if IMF doesn't lend in the first period:

$$\beta D^1_{low} + D^2_{high} > \beta D^1_{high} + D^2_{high}$$

3. If both countries threaten the default and are granted the lending, then payoffs are the same for both cases.

Exercise 5

Suppose two types of activities:[4]

- *With immediate costs: the cost is realized when the activity is performed, while a benefit comes later (studying for an exam, exercising)*
- *With immediate benefits: the benefit is realized when the activity is performed, while a cost comes later (eating chocolate ice cream, gambling)*

And two types of present-biased agents:

- *naive (N): do not realize the time inconsistency; think that future selves will implement the plan of the current self. They choose a strategy* $\mathbf{s}^n \equiv \left(s^n_1, s^n_2, \ldots, s^n_T\right)$ *that satisfies for all* $t < T$ $s^n_t = Yes$ *if and only if* $U^t(t) \geq U^t(\tau)$ *for all* $\tau > t$*; where* Yes *means that the activity is completed in a given period.*
- *sophisticated (S) do realize the time inconsistency; solve their problem by using subgame-perfect equilibrium. They choose a strategy a strategy* $\mathbf{s}^s \equiv \left(s^s_1, s^s_2, \ldots, s^s_T\right)$ *that satisfies for all* $t < T$ $s^s_t = Yes$ *if and only if* $U^t(t) \geq U^t\left(\tau'\right)$ *where* $\tau' \equiv \min_{\tau > t} \left\{\tau \mid s^s_\tau = Yes\right\}$.

Both of them compare the timing of an action of both types to the timing of time consistent (TC) agents for whom $\beta = 1$ *and time inconsistent, with* $\beta < 1$*. Let's assume:*

- *The activity must be performed exactly once in one of the time periods* $1, \ldots, T$.
- *Let* $\delta = 1$ *(for simplicity)*
- *Preferences in any period* $t \in \{1, .., T\}$ *:*

$$U_t = u_t + \beta\left[u_{t+1} + \ldots + u_T\right]$$

[4] This exercise is inspired by the article "Doing It Now or Later" by Ted O'Donoghue and Matthew Rabin (American Economic Review, 1999)

- *When evaluating timing of the activity in period t, if the person completes the activity in period $\tau \geq t$*
  - *for activities with immediate costs*

$$U_t = \begin{cases} \beta v_\tau - c_\tau & \text{if } \tau = t \\ \beta v_\tau - \beta c_\tau & \text{if } \tau > t \end{cases}$$

  - *for activities with immediate rewards*

$$U_t = \begin{cases} v_\tau - \beta c_\tau & \text{if } \tau = t \\ \beta v_\tau - \beta c_\tau & \text{if } \tau > t \end{cases}$$

1. *Suppose costs are immediate, $T = 4$, $\beta = \frac{1}{2}$ for naifs and sophisticates. Let $\mathbf{v} = (\bar{v}, \bar{v}, \bar{v}, \bar{v})$, and $\mathbf{c} = (3, 5, 8, 13)$. When the naif, sophisticate and time-consistent agent completes the activity?*
2. *Suppose rewards are immediate, $T = 4$, $\beta = \frac{1}{2}$ for naifs and sophisticates. Let $\mathbf{v} = (3, 5, 8, 13)$ and $\mathbf{c} = (0, 0, 0, 0)$. When the naif, sophisticate and time-consistent agent completes the activity?*
3. *How would you write down utility function for a given period if $\delta < 1$?*

1. $T = 4$, $\beta = \frac{1}{2}$ for naifs and sophisticates, $\mathbf{v} = (\bar{v}, \bar{v}, \bar{v}, \bar{v})$, and $\mathbf{c} = (3, 5, 8, 13)$

1. Payoff and expected payoffs of a naif if an activity is completed in a given period τ:

$$\text{If } \tau = 1:$$
$$U_1 = \frac{1}{2}\bar{v} - 3 \quad U_2 = \frac{1}{2}\bar{v} - 2.5$$
$$U_3 = \frac{1}{2}\bar{v} - 4 \quad U_4 = \frac{1}{2}\bar{v} - 6.5$$
$$\Longrightarrow \text{naif will postpone activity to period 2}$$
$$\text{If } \tau = 2:$$
$$U_2 = \frac{1}{2}\bar{v} - 5 \quad U_3 = \frac{1}{2}\bar{v} - 4$$
$$U_4 = \frac{1}{2}\bar{v} - 6.5$$
$$\Longrightarrow \text{naif will postpone activity to period 3}$$
$$\text{If } \tau = 3:$$
$$U_3 = \frac{1}{2}\bar{v} - 8 \quad U_4 = \frac{1}{2}\bar{v} - 6.5$$
$$\Longrightarrow \text{naif will postpone activity to period 4.}$$

   In result, a naif will procastrinate until the last period $T = 4$.

2. A sophisticate can solve the problem using backward induction and find a SPE.

Payoff of a sophisticate if an activity is completed in a given period τ :.

$$\text{If } \tau = 3:$$
$$U_3 = \frac{1}{2}\bar{v} - 8 \quad U_4 = \frac{1}{2}\bar{v} - 6.5$$
$$\text{If } \tau = 2:$$
$$U_2 = \frac{1}{2}\bar{v} - 5 \quad U_4 = \frac{1}{2}\bar{v} - 4$$
$$\text{If } \tau = 1:$$
$$U_1 = \frac{1}{2}\bar{v} - 3 \quad U_2 = \frac{1}{2}\bar{v} - 2.5$$
$$\implies \text{Knowing all the payoffs, a sophisticate will complete the activity in period 2.}$$

3. Time-consistent agent will complete the activity in the first period.

2. $T = 4$, $\beta = \frac{1}{2}$ for naifs and sophisticates, $\mathbf{v} = (3, 5, 8, 13)$ and $\mathbf{c} = (0, 0, 0, 0)$.

1. Payoff of a naif if activity performed in a given period τ:

$$\text{If } \tau = 1:$$
$$U_1 = 3 \quad U_2 = 2.5 \quad U_3 = 4 \quad U_4 = 6.5$$
$$\implies \text{Naif wants to postpone to period 4}$$
$$\text{If } \tau = 2:$$
$$U_2 = 5 \quad U_2 = 4 \quad U_4 = 6.5$$
$$\implies \text{Naif wants to postpone to period 4}$$
$$U_3 = 8 \quad U_4 = 6.5$$
$$\implies \text{A naif would complete the activity in the third period}$$

2. Payoff of a sophisticate if an activity performed in a given period τ:

$$\text{if } \tau = 3:$$
$$U_3 = 8 \quad U_4 = 6.5$$
$$\text{if } \tau = 2:$$
$$U_2 = 5 \quad U_3 = 4 \quad U_4 = 6.5$$
$$\text{If } \tau = 1:$$
$$U_1 = 3 \quad U_2 = 2.5 \; U_3 = 4 \quad U_4 = 6.5$$

This case is more interesting. In the third period a sophisticate would give in and get the reward. Knowing that, a sophisticate in the second period would not wait and get the reward in the second period (and get payoff 5>4). Knowing that, a sophisticate in the first period would get the reward in the first period (3>2.5). Interestingly, a sophisticate get the worse payoff than a naif in this context.

3. Time consistent would wait until the last period.

3.

If $\delta < 1$, if the person completes the activity in period $\tau \geq t$

- for activities with immediate costs

$$U_t = \begin{cases} \beta v_\tau - c_\tau & \text{if } \tau = t \\ \delta^\tau(\beta v_\tau - \beta c_\tau) & \text{if } \tau > t \end{cases}$$

- for activities with immediate rewards

$$U_t = \begin{cases} v_\tau - \beta c_\tau & \text{if } \tau = t \\ \delta^\tau(\beta v_\tau - \beta c_\tau) & \text{if } \tau > t \end{cases}$$

Exercise 6

Consider the following game where player 1 is a long-run player and player 2 is a short-run player:[5]

		P2		
		L	M	R
P1	U	1,3	3,4	-1,0
	D	0,3	2,2	-1,0

1. *Find all Nash equilibria of this game.*
2. *What is the worst dynamic equilibrium of this game?*
3. *Find the pure and mixed Stackelberg equilibrium in which the long-run player moves first.*
4. *What is the best dynamic equilibrium and for what discount factor is it attainable?*

We find Nash equilibria by looking at the best responses for each player:

$$BR_1(L) = U \;\; BR_1(M) = U \;\; BR_1(R) = \Delta\{U, D\}$$
$$BR_2(U) = M \;\; BR_2(D) = L$$

We have one PSNE: (U,M).
In order to find MSNE we start with indifference condition for P2. Suppose player 1 plays U with probability α_1, and D with probability $1 - \alpha_1$. Then we have:

$$3\alpha_1 + 3(1 - \alpha_1) = 4\alpha_1 + 2(1 - \alpha_1)$$
$$\implies \alpha_1 = \frac{1}{2}$$

[5] This exercise is inspired by the article "Repeated Games with Long-Run and Short-Run Players" by D. Fudenberg, D.M. Kreps, E. Maskin (The Review of Economic Studies, 1990)

Note that we don't take into account the strategy R as it is strictly dominated by both strategies L and M. What about P1? Denote β_1 as probability of P2 playing L, β_2 playing M and $1-\beta_1-\beta_2$ playing R. Then we have:

$$\beta_1 + 3\beta_2 - \beta_1 - \beta_2 = 2\beta_2 - \beta_1 - \beta_2$$
$$\implies \beta_3 = 1$$

Note that only when P2 plays R player 1 is indifferent between two strategies. Otherwise, she prefer to play U.

However, even if indifference conditions can be derived there is no MSNE as for P1 strategy U weakly dominates strategy U. If $\sigma_1(U) = 1$, then $\sigma_2(M)$. There is unique PSNE, which is (U,M).

The worst dynamic equilibrium lies between minmax payoff and Nash eq. payoff. One method is to check whether the minmax is the same as Nash:

$$minmax = \min_{\alpha^2} \max_{\alpha^1} u^1\left(a^1, \alpha^2\right) = -1$$

Here minmax and Nash equilibrium do not coincide. We know from the lecture that the worst dynamic equilibrium payoff is "the constrained minmax":

$$\underline{v}^1 = \min_{\alpha^2 \in BR^2(\alpha^1)} \max_{\alpha^1} u^1\left(a^1, \alpha^2\right)$$

Hence, in searching for $\underline{v}^1$ we constrain ourselves only to strategies which were played by the short-term player as best responses here are L and M. Definitely smaller payoffs for P1 gives a strategy L. If P2 plays L, the best response of P1 us to play U. Hence, the worst dynamic equilibrium is (U,L) yielding payoffs (1,3).

Notice that the only strategy for P1 that makes sense to precommit is strategy U. $BR_2(U) = M$, hence the pure Stackelberg would be the same as Nash equilibrium. Actually, we do not need a preocommitment in this game to achieve better equilibria than Nash.

What about mixed Stackelberg?

Let's see what are the best responses of P2 given the value of $\sigma_1(U)$:

$$BR_2(\sigma_1(U)) = \begin{cases} L\ if\ \sigma_1(U) < 1/2 \\ \Delta\{L, M\}\ if\ \sigma_1(U) = 1/2 \\ M\ if\ \sigma_1(U) > 1/2 \end{cases}$$

Assume the tie-breaking rule in which, if P2 indifferent, she would play in favour of P1. As we see that P1 prefer P2 to play M, the payoff function would then be:

$$u_1(\sigma_1(U) > 1/2, M) = 3\sigma_1(U) + 2(1 - \sigma_1(U)) = \sigma_1(U) + 2$$

The larger the probability of P1 playing U, the higher expected payoff for her. Hence, mixed Stackelberg equilibrium will be the same as pure precommitment and Nash equilibrium, yielding payoff 3 for the long-run player.

The best dynamic equilibrium lays between pure and mixed Stackelberg, but in this example they coincide so it is (U,M). Let's see whether we need any constraint on the discount factor to attain this equilibrium:

$$3 \geq (1-\delta)1 + 2\delta$$
$$2 \geq \delta$$
$$\delta \in [0,1]$$

For any value of δ we will achieve best dynamic equilibrium.

Exercise 7

Consider a lottery in which you can win 1000 EUR with probability 1/2 and lose with probability 1/2. Before making a decision you are approached by an insurer who offers you an insurance removing the risk costs of 500 EUR.[6]

1. *At what level of wealth will you be indifferent between taking the gamble or paying the insurance? Consider two utility functions: $u(Y) = Y^{0.5}$ and $u(Y) = ln(Y)$.*
2. *How much will you pay to avoid this risk if your current level of wealth is 10,000 EUR? How much would you pay if your level of wealth is 1 million EUR? Did you expect that the premium you were willing to pay would increase/decrease? Why?*

In the first part we need to find certainty equivalent. Consider firstly $u(Y) = Y^{0.5}$:

$$0.5u(Y+1000) + 0.5u(Y-1000) = u(Y-500)$$
$$(Y+1000)^0.5 + (Y-1000)^0.5 = 2(Y-500)^0.5$$
$$[(Y+1000)^0.5 + (Y-1000)^0.5]^2 = 4Y - 1000$$

Don't worry, you won't need to solve unsolvable-by-hand exercises during the exam. We need an Y which satisfies the following equation. We can infer that the wealth can't be lower than 250 (then the LHS would be negative).

Consider now $u(Y) = ln(Y)$:

$$0.5ln(Y+1000) + 0.5ln(Y-1000) = ln(Y-500)$$
$$ln((Y-1000)(Y+1000)) = ln(Y-500)^2$$
$$(Y-1000)(Y+1000) = (Y-500)^2$$
$$Y^2 - 1000^2 = Y^2 - 1000Y + 500^2$$
$$1000Y = 2500 + 100000$$
$$10Y = 25 + 1000$$
$$Y = 2.5 + 100 = 102.5$$

[6] This exercise is a modification of one of exercises found at prof. John Donaldson's website from the Columbia Business School.

In the second part we need to find risk premium, p. Let's start with the first utility function:

$$0.5(Y+1,000)^0.5 + 0.5(Y-1,000)^0.5 = (Y-p)^0.5$$

$$\text{if } Y = 10,000:$$

$$\Longrightarrow p = 25.0628$$

$$\text{if } Y = 1,000,000:$$

$$\Longrightarrow p = 0.25$$

For the utility function $u(Y) = Y^0.5$ the risk premium decreases with the level of wealth, which is in line with expectations (decreasing marginal utility of wealth).
For the logarithmic utility function we have:

$$0.5lnY + 1,000) + 0.5ln(Y-1,000) = ln(Y-p)$$

$$\text{if } Y = 10,000:$$

$$\Longrightarrow p = 50.126$$

$$\text{if } Y = 1,000,000:$$

$$\Longrightarrow p = 0.5$$

Similarly, the risk premium decreases with the level of wealth. Observe that for this utility function the risk premium is higher for both level of wealth than in the case of the previous utility function. Can we infer it from the chracteristics of both utility functions?